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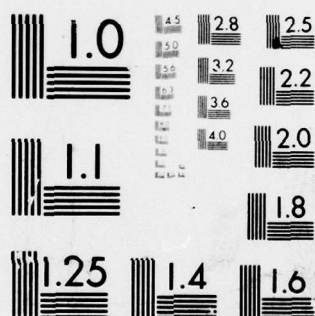
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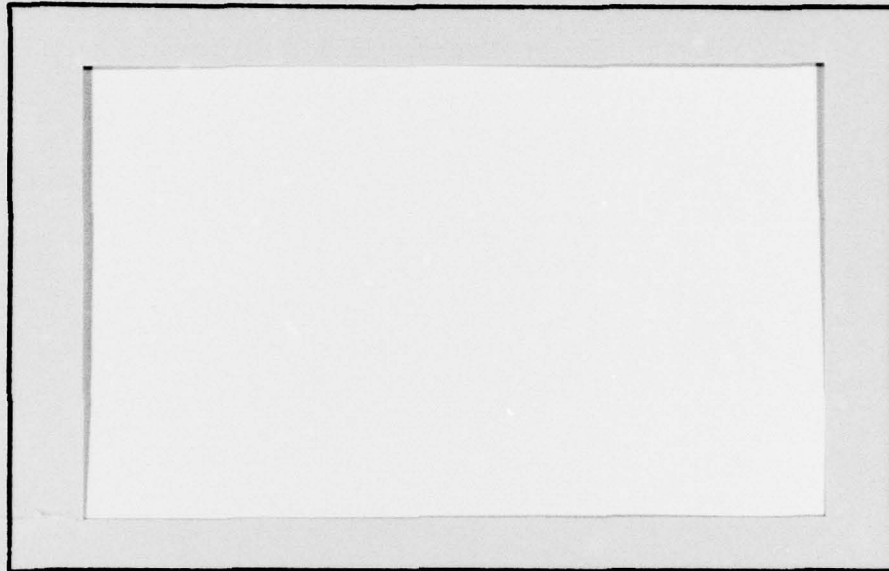
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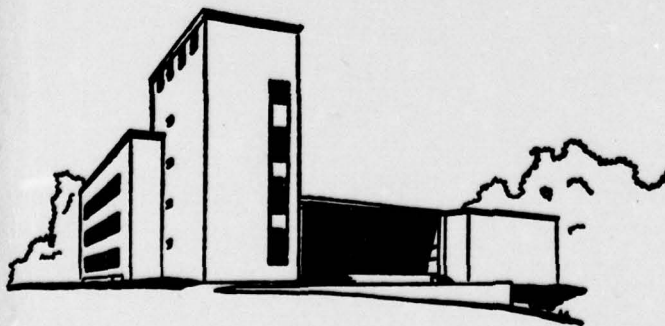
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# ① LEVEL II

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⑥ Degree Sequences of Random Graphs.

by

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Carnegie-Mellon University

⑪ November 1979

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## Degree Sequences of Random Graphs

by Gerard Cornuejols

### ABSTRACT

A property is said to be satisfied by almost all graphs on  $n$  vertices if the fraction of the graphs that satisfy it goes to 1 as  $n$  goes to  $\infty$ . For almost all graphs on  $n$  vertices the smallest vertex-degree is  $n/2 - \sqrt{(n \log n)/2} + o(\sqrt{n})$ . More generally we estimate the values in the tails of the degree sequence. We also show that the degree sequence  $d_1 \geq d_2 \geq \dots \geq d_n$  of almost all graphs on  $n$  vertices satisfies  $d_1 \neq d_2 \neq \dots \neq d_k$  and  $d_{n-k} \neq \dots \neq d_{n-1} \neq d_n$  for all  $k \leq n^\alpha$ ,  $\alpha < 1/4$ .

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## 1. Introduction

In this paper we consider the graphs without loops or multiple edges whose vertices are labeled  $v_1, \dots, v_n$ . There are  $2^{n(n-1)/2}$  such labeled graphs. Note that two different labeled graphs may be isomorphic. Given a graphical property a typical question in graphical enumeration is to compute the fraction of the labeled graphs on  $n$  vertices that satisfy it. When this fraction goes to 1 as  $n$  goes to  $\infty$  we say that the graphical property is satisfied by almost all labeled graphs. This notion was introduced by Erdos and Reyni [4].

Graph theorists have traditionally spent much effort in enumerating non isomorphic graphs rather than labeled graphs. A "metatheorem" of Harary and Palmer [9] states that a graphical property holds for almost all non isomorphic graphs if and only if it holds for almost all labeled graphs. In view of this theorem the remainder of this paper will only be concerned with labeled graphs. We will often simply refer to labeled graphs as graphs for brevity.

There is a useful equivalent probabilistic view of the notion of a graphical property satisfied by almost all labeled graphs. Assume that all labeled graphs on  $n$  vertices are equally likely to occur; labeled graphs drawn randomly from this uniform probability distribution are denoted by  $G_n$ . It is easily checked that, in a random graph  $G_n$ , each of the possible  $n(n-1)/2$  edges  $(v_i, v_j)$ ,  $i \neq j$ , occurs independently with probability  $1/2$ . Given some graphical property  $A_n$ , we denote by  $p_n = \Pr[A_n]$  the probability that a random graph  $G_n$  satisfies the property  $A_n$ . Note that  $p_n$  is the fraction of the labeled graphs on  $n$  vertices that satisfy  $A_n$ . So  $A_n$  holds for almost all labeled graphs if and only if the probability  $p_n$  goes to 1 as  $n$  goes to  $\infty$ . An obvious generalization of this notion of random graphs is obtained when each of the possible  $n(n-1)/2$  edge occurs independently with probability  $p$ , where

$0 < p < 1$ . A random graph with such a probability distribution will be denoted by  $G_n(p)$ . With our notation  $G_n(1/2) = G_n$ . We say that a property  $A_n$  holds for almost all graphs  $G_n(p)$  if and only if  $p_n = \Pr[A_n]$  goes to 1 as  $n$  goes to  $\infty$ . Let us mention also the notion of random graphs used by Erdos and Renyi [5]. They assume that all  $\binom{n(n-1)/2}{m}$  labeled graphs with  $n$  vertices and  $m$  edges are equally likely to occur. Random graphs drawn from this distribution are denoted by  $G_{n,m}$ .

In [5], Erdos and Renyi mentioned the study of the vertex-degrees in random graphs as an interesting question. As a first step they proved

Theorem. Let  $d_1(G_{n,m})$  be the largest vertex-degree in  $G_{n,m}$  and  $d_n(G_{n,m})$  the smallest. If  $m$  is a function of  $n$  such that  $\lim_{n \rightarrow \infty} \frac{m}{n \log n} = \infty$ , then for any  $\epsilon > 0$   $\lim_{n \rightarrow \infty} \Pr \left[ \frac{d_1(G_{n,m})}{d_n(G_{n,m})} - 1 < \epsilon \right] = 1$ .

In [10], Ivchenko gave the limiting distribution of the vertex-degrees in  $G_n(p)$  in the cases when  $p \rightarrow 0$  as  $n \rightarrow \infty$  and when  $p \rightarrow 1$  as  $n \rightarrow \infty$ . However his approach is limited by the conditions on  $p$ . Recently Erdos and Wilson [6] made a new contribution to the study of the vertex-degrees with the following result.

Lemma. Almost all graphs  $G_n$  have a unique vertex of largest degree.

In this paper we give further properties of the vertex-degrees in random graphs. These properties are generalizations of the results of Erdos and Wilson [6]. Consider a random graph  $G_n(p)$  where  $p$  is a constant probability,  $0 < p < 1$ . Let  $q = 1 - p$ . In the sequel we will write

$f(n) = o(g(n))$  to mean that  $\frac{f(n)}{g(n)} \rightarrow 0$  as  $n \rightarrow \infty$ ;  $f(n) = O(g(n))$  will mean

that  $\frac{f(n)}{g(n)} \rightarrow c$  as  $n \rightarrow \infty$  for some constant  $c > 0$  and  $f(n) \sim g(n)$  will mean that  $\frac{f(n)}{g(n)} \rightarrow 1$  as  $n \rightarrow \infty$ , where  $f$  and  $g$  are real valued functions defined on the set of positive integers and  $g$  never takes the value 0.

Given a graph on  $n$  vertices, we denote by  $d_1 \geq d_2 \geq \dots \geq d_n$  its degree sequence. In section 2 we prove that for almost all graphs  $G_n(p)$

$$(1) \quad d_k = np + \sqrt{2pq n \log n} + o(\sqrt{n})$$

for any  $k \leq (\log n)^c$  where  $c$  is some positive constant. In particular, when  $p = \frac{1}{2}$  and  $k = 1$  we have

$$(2) \quad d_1 = n/2 + \sqrt{(n \log n)/2} + o(\sqrt{n})$$

for almost all graphs. This corrects an error that appeared in the lemma of Erdos and Wilson [6].

(1) is a property of the tail of the degree sequence in the sense that it deals with  $d_k$  where  $k$  is small compared to  $n$ . Another property of the tail of the degree sequence proved in section 2 is that almost all graphs  $G_n(p)$  satisfy

$$(3) \quad d_k - d_{k+1} > \ell$$

for all positive integers  $k$  and  $\ell$  such that  $k \leq n^\alpha$  and  $\ell \leq n^\beta$  where  $2\alpha + \beta < \frac{1}{2}$ . By setting  $k = \ell = 1$  in (3) it follows that almost all graphs  $G_n(p)$  have a unique vertex of largest degree.

Recently the study of asymptotic results in random graphs has received a great deal of attention as a tool for analyzing some hard optimization problems on graphs. See, e.g. Grimmett and McDiarmid [8], Posa [14], Karp [11], Chvatal [1], [2], McDiarmid [12], Pulleyblank [15]. Our motivation

for the study of the degree sequences of random graphs also found its roots in analyzing an optimization problem on graphs: given a graph  $G = (V, E)$  and a positive integer  $k < n = |V|$ , find  $k$  vertices of  $G$  that cover the maximum number of edges. This edge covering problem is analyzed in Cornuejols, Nemhauser and Wolsey [3]. It can be formulated as the following integer linear program:

$$\begin{aligned}
 z &= \max \sum_{e \in E} x_e \\
 x_e &\leq y_u + y_v \quad \text{for all } e = (u, v) \in E \\
 \sum_{v \in V} y_v &= k \\
 x_e &= 0 \text{ or } 1 \quad \text{for all } e \in E \\
 y_v &= 0 \text{ or } 1 \quad \text{for all } v \in V.
 \end{aligned}$$

The linear programming relaxation obtained by replacing the integer valued variables by real variables in the interval  $[0, 1]$  gives an upper bound  $z_L$  on the value  $z_I$ . A lower bound  $z_H$  can be obtained by applying the following simple heuristic: pick  $k$  vertices with largest degrees in the graph  $G$  and let  $z_H$  be the number of edges that they cover. It is shown in [3] that  $z_L = z_I = z_H$  whenever the graph  $G$  satisfies condition (3) with  $k = k$ . Thus the heuristic and the relaxation find an optimal solution to the edge covering problem for almost all graphs  $G_n(p)$  and all  $k < n^\alpha$ ,  $\alpha < 1/6$ .

## 2. Vertex-degrees in random graphs

The vertex-degrees in  $G_n(p)$  are identically distributed and have a binomial distribution  $B(n-1, p)$ . Furthermore note that there is a small positive correlation between the degrees of any two vertices  $v_i$  and  $v_j$

since both degrees are increased by 1 if the edge  $(v_i, v_j)$  occurs in the graph, by 0 if it does not.

The general approach in this section will be, using the normal approximation to the binomial distribution, to solve a simplified problem with no correlation, then to introduce the correlation. It is interesting to parallel this approach with the work of Moon [13] on the maximum score in random tournaments. (In this problem there is a small negative correlation between the scores of any two players.)

Our first theorem will give the range of the degree sequence in a random graph  $G_n(p)$ . It is a consequence of the following lemma.

Lemma 1. Let  $X_1, \dots, X_n$  be  $n$  identically distributed random variables having the binomial distribution  $B(n-1, p)$ . Then the values of  $X_i$ ,  $i = 1, \dots, n$ , all lie in the interval  $[np - \sqrt{2pq n \log n}, np + \sqrt{2pq n \log n}]$  with a probability that goes to 1 as  $n$  goes to  $\infty$ .

Proof: Consider the normalized random variables  $X_i^* = \frac{X_i - (n-1)p}{\sqrt{pq(n-1)}}$ . Let

$X_{[1]}^* = \max_{i=1, \dots, n} X_i^*$ . By Boole's inequality, for any value  $x_n$ ,

$$(4) \quad \Pr[X_{[1]}^* > x_n] \leq n \Pr[X_1^* > x_n].$$

By the theorem of large deviations, Feller [7], for any fixed  $\epsilon > 0$  and for all  $x_n \rightarrow \infty$  such that  $x_n = o(n^{1/6})$ ,

$$(5) \quad \Pr[X_1^* > x_n] \leq \frac{1 + \epsilon}{x_n \sqrt{2\pi}} e^{-x_n^2/2}.$$

Choose

$$(6) \quad x_n = \sqrt{2 \log n}.$$

Then (4)-(6) yield  $\Pr[X_{[1]}^* > \sqrt{2 \log n}] \leq \frac{1+\epsilon}{2\sqrt{\pi \log n}}$ . So

$\Pr[X_{[1]}^* > \sqrt{2 \log n}] \rightarrow 0$  as  $n \rightarrow \infty$ , i.e., with a probability that goes to 1 as  $n$  goes to  $\infty$ ,  $\frac{X_{[1]} - (n-1)p}{\sqrt{pq(n-1)}} \leq \sqrt{2 \log n}$ . The upper bound

$X_{[1]} \leq (n-1)p + \sqrt{2pq(n-1) \log n}$  follows. Keeping only the highest order terms we get  $X_{[1]} \leq np + \sqrt{2pq n \log n}$ . A similar analysis of

$X_{[n]}^* = \min_{i=1, \dots, n} X_i^*$  yields the lower bound  $np - \sqrt{2pq n \log n}$ .

**Theorem 1.** For almost all graphs  $G_n(p)$  the vertex-degrees all lie in the interval  $[np - \sqrt{2pq n \log n}, np + \sqrt{2pq n \log n}]$ .

**Proof:** This is a direct consequence of Lemma 1 since the vertex-degrees in  $G_n(p)$  are identically distributed with the binomial distribution  $B(n-1, p)$ .

Note that the range of the degree sequence can be slightly tightened by choosing  $x_n = \sqrt{2 \log n} - (1-\epsilon) \frac{\log \log n}{\sqrt{8 \log n}}$  in equation (6) of the proof of Lemma 1. This yields

**Proposition 1.** For any  $\epsilon > 0$  and for almost all graphs  $G_n(p)$

$$d_1 < np + \sqrt{2pq n \log n} - (1-\epsilon) \sqrt{\frac{pq n}{8 \log n}} \log \log n \quad \text{and}$$

$$d_n > np - \sqrt{2pq n \log n} + (1-\epsilon) \sqrt{\frac{pq n}{8 \log n}} \log \log n.$$

A consequence of Theorem 1 is that, for almost all graphs  $G_n(p)$ , the vertex-degrees are integers from a range of at most  $2\sqrt{2pq n \log n}$  values. Thus

**Corollary 1.** The degree sequence of almost all graphs  $G_n(p)$  contains a

subsequence of at least  $\sqrt{\frac{n}{8pq \log n}}$  identical values.

The next theorem provides the values of the  $k$  largest vertex-degrees in  $G_n(p)$ . In order to prove the theorem we need the following lemma for independent random variables.

**Lemma 2.** Let  $X_1, \dots, X_m$  be  $m$  identically and independently distributed random variables having the binomial distribution  $B(\ell, p)$ . Assume that  $\frac{\ell}{(\log m)^3} \rightarrow \infty$  as  $m \rightarrow \infty$ . Let  $X_{[k]}$  be the  $k^{\text{th}}$  order statistic, i.e.,  $X_{[1]} \geq \dots \geq X_{[k]} \geq \dots \geq X_{[m]}$ . Then with a probability that goes to 1 as  $n$  goes to  $\infty$ ,

$$(7) \quad X_{[k]} > \ell p + \sqrt{p q \ell} \left( \sqrt{2 \log m} - \frac{(c+2) \log \log m}{\sqrt{2 \log m}} \right)$$

for all  $k \leq (\log m)^c$  where  $c$  is a positive constant;

$$(8) \quad X_{[k]} > \ell p + \sqrt{a p q \ell \log m}$$

for all  $k \leq n^\alpha$  where  $\frac{a}{2} + \alpha < 1$ .

**Proof.** Consider the normalized random variables  $X_i^* = \frac{X_i - \ell p}{\sqrt{p q \ell}}$ . By

Boole's inequality, for any value  $x_m$ ,

$$(9) \quad \Pr[X_{[k]}^* \leq x_m] \leq \binom{m}{k-1} \Pr[X_1^* \leq x_m]^{m-k-1}$$

To bound the right side of (9), note that

$$(10) \quad \binom{m}{k-1} \leq m^k = e^{k \log m}$$

and that  $\Pr[X_1^* \leq x_m] = 1 - \Pr[X_1^* > x_m] \leq 1 - \frac{1-\epsilon}{x_m \sqrt{2\pi}} e^{-x_m^2/2}$  for all  $\epsilon > 0$  and  $x_m = o(\ell^{1/6})$ , by the theorem of large deviations. Since  $1 - y \leq e^{-y}$  for all values of  $y$ ,

$$(11) \quad \Pr[X_1^* \leq x_m]^{m-k+1} \leq e^{-(m-k+1)} \frac{1-\epsilon}{x_m \sqrt{2\pi}} e^{-x_m^2/2}$$

To prove the inequality (7) when  $k \leq (\log m)^c$ , take

$$x_m = \sqrt{2 \log m} - \frac{(c+2) \log \log m}{\sqrt{2 \log m}}. \quad \text{Then (9)-(11) yield, when } m \rightarrow \infty,$$

$$\Pr[X_{[k]}^* \leq x_m] \leq e^{(\log m)^{c+1}} e^{-\frac{1-\epsilon}{\sqrt{4\pi}} (\log m)^c + \frac{3}{2}}$$

Thus  $X_{[k]}^* > x_m$  with a probability that goes to 1 as  $m$  goes to  $\infty$ . (7) follows.

Similarly, to prove the inequality (8) when  $k \leq n^\alpha$ , take

$$x_m = \sqrt{a \log m}. \quad (9) - (11) \text{ yield, when } m \rightarrow \infty,$$

$$\Pr[X_{[k]}^* \leq x_m] \leq e^{m^\alpha \log m} e^{-\frac{1-\epsilon}{\sqrt{2\pi a \log m}} m^{1-a/2}}$$

Thus  $\Pr[X_{[k]}^* \leq x_m] \rightarrow 0$  as  $m \rightarrow \infty$  for  $\alpha < 1 - \frac{a}{2}$ .

**Theorem 2.** Almost all graphs  $G_n(p)$  satisfy the following two properties

$$(12) \quad d_k = np + \sqrt{2pq n \log n} - o(\sqrt{n})$$

for all  $k \leq (\log n)^c$  where  $c$  is a positive constant;

$$(13) \quad np + \sqrt{apqn \log n} < d_k < np + \sqrt{2pq n \log n}$$

for all  $k \leq n^\alpha$  where  $\frac{a}{2} + \alpha < 1$ .

**Proof.** Consider an arbitrary subset  $S_n$  of the vertices of  $G_n(p)$  that has cardinality  $m = \lfloor \frac{n}{(\log n)^2} \rfloor$ . For a vertex  $v_i$  in  $S_n$  let  $a(v_i)$  be the number of vertices not in  $S_n$  that are adjacent to  $v_i$ . Let  $a_k$  be the  $k^{\text{th}}$  largest  $a(v_i)$ . Finally let  $s(v_i)$  be the degree of vertex  $v_i$  in the subgraph induced on  $S_n$ . Note that

$$(14) \quad d_k \geq a_k + \min_{v_i \in S_n} s(v_i) .$$

Applying Theorem 1 to the random subgraph induced on  $S_n$  yields

$$(15) \quad \min_{v_i \in S_n} s(v_i) > mp - \sqrt{2pqm \log m}$$

for almost all graphs  $G_n(p)$ . Since the random variables  $a(v_i)$  are independent and identically distributed with the distribution  $B(n-m, p)$ , Lemma 2 can be applied to bound  $a_k$ .

For almost all graphs  $G_n(p)$  and all  $k \leq (\log n)^c$

$$(16) \quad a_k > (n-m)p + \sqrt{2pq(n-m)} (\sqrt{\log m} - o(1)).$$

(14)-(16) imply that  $d_k > np + \sqrt{2pq n \log n} - o(\sqrt{n})$ , where only the highest order terms have been kept in the right side. Equality (12) of Theorem 2 follows from this lower bound and the upper bound of Theorem 1.

Similarly (13) is a consequence of (14), (15), (8) and Theorem 1.

Theorems 1 and 2 state that some properties of the degree sequence in  $G_n(p)$  have a probability that goes to 1 as  $n$  goes to  $\infty$ . They are proved by lower bounding the probability by a function of  $n$  and showing that the bounds are asymptotically equal to 1. These bounds provide interesting information for medium size random graphs also, as shown in Table 1.

$n$	more than 95% of the graphs have a range smaller than	range of Theorem 1	more than 95% of the graphs have a range larger than
100	[32, 67]	[35, 64]	[40, 59]
1,000	[435, 564]	[441, 558]	[457, 542]
10,000	[4771, 5228]	[4785, 5214]	[4834, 5165]

The last theorem will show that in the tails of the degree sequence the vertex-degrees are all distinct. Actually it provides a lower bound on the difference between consecutive values in the tails of the degree sequence. To prove it we need a technical lemma.

Lemma 3. Let  $n$  be a positive integer,  $0 < p < 1$ ,  $q = 1 - p$ ,  $0 < \beta < \frac{1}{2}$  and  $0 < a < 2$ . Define  $\delta$  and  $h$  to be the integer parts of  $np$  and  $\sqrt{apq n \log n}$  respectively, and  $p_n = n^{2+\beta} \sum_{i=\delta+h}^n \left[ \binom{n}{i} p^i q^{n-i} \right]^2$ . Assume that  $a > \beta + 3/2$ . Then  $p_n \rightarrow 0$  as  $n \rightarrow \infty$ .

Proof. In the sum  $\sum_{i=\delta+h}^n \left[ \binom{n}{i} p^i q^{n-i} \right]^2$  bound each of the first  $h$  terms by  $\left[ \binom{n}{\delta+h} p^{\delta+h} q^{n-\delta-h} \right]^2$  and each of the remaining terms by  $\left[ \binom{n}{\delta+2h} p^{\delta+2h} q^{n-\delta-2h} \right]^2$ . Thus

$$p_n \leq n^{2+\beta} \left( \left[ \binom{n}{\delta+h} p^{\delta+h} q^{n-\delta-h} \right]^2 + nq \left[ \binom{n}{\delta+2h} p^{\delta+2h} q^{n-\delta-2h} \right]^2 \right).$$

When  $n$  is large, an estimate of the binomial probabilities is available, see e.g. Feller [7].

$$\binom{n}{\delta+tr} p^{\delta+tr} q^{n-\delta-tr} = \frac{1}{\sqrt{2\pi p q n}} e^{\frac{-r^2}{2p q n} + \rho_r}$$

where  $|\rho_r| < \frac{r^2}{(p q n)^2} + \frac{2r}{p q n}$  so

$$p_n \leq n^{2+\beta} \left( \frac{h}{2\pi p q n} e^{-\frac{h^2}{p q n}} + \frac{nq}{2\pi p q n} e^{-\frac{4h^2}{p q n}} \right) (1 + o(1))$$

$$\leq \frac{\sqrt{a \log n}}{2\pi \sqrt{p q}} n^{3/2 + \beta - a} + \frac{1}{2np} n^{2 + \beta - 4a} (1 + o(1)).$$

Now  $p_n \rightarrow 0$  as  $n \rightarrow \infty$  if  $3/2 + \beta - a < 0$  and  $2 + \beta - 4a < 0$ . Both inequalities follow from the assumptions of the lemma.

**Theorem 3.** Almost all graphs  $G_n(p)$  satisfy

$$d_k - d_{k+1} > \ell$$

for all positive integers  $k$  and  $\ell$  such that  $k \leq n^\alpha$  and  $\ell \leq n^\beta$  where  $2\alpha + \beta < \frac{1}{2}$ .

**Proof.** Given a vertex  $v$  of  $G_n(p)$  and a set of  $m$  vertices that does not contain  $v$ ,  $b(i; m, p) = \binom{m}{i} p^i q^{m-i}$  is the probability that  $v$  is incident with exactly  $i$  of the  $m$  vertices. The probability that two given vertices have respective degrees  $s$  and  $s + t$  is

$$p b(s-1; n-2, p) b(s+t-1; n-2, p)$$

$$+ q b(s; n-2, p) b(s+t; n-2, p) .$$

Thus, given the positive integers  $s$  and  $\ell$ , the probability that two given vertices of  $G_n(p)$  have respective degrees  $s$  and  $s + t$  for some  $t$ ,  $0 \leq t \leq \ell$  is

$$\begin{aligned} P(s, \ell) = & p b(s-1; n-2, p) \sum_{t=0}^{\ell} b(s+t-1; n-2, p) \\ & + q b(s; n-2, p) \sum_{t=0}^{\ell} b(s+t; n-2, p) \end{aligned}$$

( $b(i; m, p)$  is assumed to be 0 if  $i > m$ ). For any positive integer  $r < n-1$  the probability that at least two vertices of  $G_n(p)$  have respective degrees  $s$  and  $s+t$  for some  $s \geq r$  and some  $t$ ,  $0 \leq t \leq \ell$ , is upper bounded by

$$\pi_n = n(n-1) \sum_{s=r}^{n-1} P(s, \ell) .$$

Thus the probability that  $d_k - d_{k+1} \leq \ell$  for positive integers  $k$  and  $\ell$  satisfying  $k \leq n^\alpha$  and  $\ell \leq n^\beta$  is at most

$$P[d_{k+1} < r] + \pi_n .$$

Now we assume that  $r$  is the integer part of  $np + \sqrt{apqn \log n}$ , where  $a > 0$  and  $\frac{a}{2} + \alpha < 1$ . Then by the property (13) in Theorem 2,  $\Pr[d_{k+1} < r] \rightarrow 0$  as  $n \rightarrow \infty$ . To bound  $\pi_n$  note that  $b(s; n, p)$  is a decreasing function of  $s$  for  $s > np$ . Thus, for all  $s \geq r$ ,

$$\begin{aligned} P(s, \ell) &\leq p(\ell+1) b(s-1; n-2, p)^2 + q(\ell+1) b(s; n-2, p)^2 \\ &\leq (\ell+1) b(s-1; n-2, p)^2 \end{aligned}$$

Therefore  $\pi_n \leq n(n-1)(\ell+1) \sum_{s=r}^{n-1} b(s-1; n-2, p)^2$ . By Lemma 3 and the assumption that  $2\alpha + \beta < \frac{1}{2}$ ,  $\pi_n \rightarrow 0$  as  $n \rightarrow \infty$ .

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<p>A property is said to be satisfied by almost all graphs on <math>n</math> vertices if the fraction of the graphs that satisfy it goes to 1 as <math>n</math> goes to <math>\infty</math>. For almost all graphs on <math>n</math> vertices the smallest vertex-degree is</p> <p><math>n/2 - \sqrt{(n \log n)/2} + o(\sqrt{n})</math>. More generally we estimate the values in the tails of the degree sequence. We also show that the degree sequence <math>d_1 \geq d_2 \geq \dots \geq d_n</math> of almost all graphs on <math>n</math> vertices satisfies <math>d_1 \neq d_2 \neq \dots \neq d_k</math> and <math>d_{n-k} \neq \dots \neq d_{n-1} \neq d_n</math> for all <math>k \leq n^\alpha</math>, <math>\alpha &lt; 1/4</math>.</p>		

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